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ically; whence finally

$$|R_n(x)| < N\sqrt{\frac{2}{\pi(2n+1)}} \tag{7}$$

for every value x_0 of x and for every $n \geqq 0$. Thus the remainder approaches 0 as n increases indefinitely, and the Fourier's series converges to $f(x)$.

The method outlined above may easily be modified so as to deal with a more general type of function $f(x)$, for instance any periodic function with two derivatives. Essentially the only novel feature of the method is the direct use of an equation like (4) to obtain an upper limit for the remainder.[1]

A CUBIC SPACE CURVE CONNECTED WITH THE TETRAHEDRON.

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In connection with the geometry of the triangle there is a well known circumconic, known as Kiepert's hyperbola,[2] which passes through the orthocenter and centroid of the triangle. In the following note an analogue of this hyperbola for the tetrahedron is given, and it is hoped that interest may thereby be aroused in the still comparatively unexplored field of the "geometry of the tetrahedron." [3]

It will be convenient to recall an interesting property of Kiepert's hyperbola—a property which may be used to define the curve and at the same time to give a simple parametric representation of the points on it. Let A_1, A_2, A_3 be the triangle and on each of the segments A_2A_3, A_3A_1, A_1A_2 describe a circle containing an angle θ; θ being in each case the angle subtended by the segment at points of that arc of the corresponding circle which is on the same side of the segment as the opposite vertex, so that θ can vary from 0° to 180°. For any given value of θ the three circles have a radical center and Kiepert's hyperbola is the locus of these radical centers. In order to find the coördinates of the radical center for a given value of θ, it is probably simplest to introduce, momentarily, rectangular axes with origin at O_1, the mid-point of A_2A_3, and with the X axis along A_2A_3. If $2a_1$ is the length of A_2A_3, the coördinates of C_1, the center of the circle through A_2A_3, are $(0, a_1 \cot\theta)$ and the radius is $a_1 \operatorname{cosec}\theta$, so that the equation is

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[1] See Lebesgue, *Leçons sur les séries trigonométriques*, Paris, 1906, pp. 37–38, and de la Vallée Poussin, *Bulletins de l'Académie royale de Belgique*, classe des sciences, 1908, pp. 193–254, in particular p. 230 *et seq.*

[2] Kiepert, *Nouvelles Annales de Mathématiques*, 1869, p. 42. Other discussions of this hyperbola are: by Brocard, in *Journal de Mathématiques Spéciales*, 1884, pp. 197–209 and 1885, pp. 12–15, 30–33, 58–64, 76–80, 104–112, 123–131; by de Longchamps, *idem*, 1886, pp. 77–79 231–235; by M'Cay, in *Mathesis*, 1887, pp. 208–220; by Laisant, in *Compte rendu . . . Association Française pour l'Avancement des Sciences*, seconde partie, 1887, pp. 113–114; by J. Casey, in his *A Treatise on the Analytical Geometry of the Point, Line, Circle and Conic Sections*, second edition, 1893, pp. 431, 442–445, 449, 452, 453; and by I. J. Schwatt, in his *A geometrical Treatment of Curves which are isogonal conjugate to a straight line with respect to a triangle.* Part 1, Boston, New York, and Chicago, [c. 1895], pp. 3–28.—EDITOR.

[3] Those interested may be referred to two papers by J. Neuberg in *Archiv der Mathematik und Physik*, 3. Reihe, vols. 16 and 18, 1910–1911.

$X^2 + Y^2 - 2a_1 Y \cot \theta - a_1^2 = 0$. The equation of the circumcircle is

$$S \equiv X^2 + Y^2 - 2a_1 Y \cot A_1 - a_1^2 = 0,$$

so that our circle may be written in the form $S + 2a_1 Y(\cot A_1 - \cot \theta) = 0$. It will now be better to introduce trilinear coördinates (x_1, x_2, x_3), x_1 signifying as usual the perpendicular distance of the point from the side A_2A_3, and so being identical with Y above. Accordingly our equation is $S + 2a_1x_1(\cot A_1 - \cot \theta) = 0$ where $S = 0$ is the equation of the circumcircle in trilinear coördinates (so written that its norm—the coefficient of $X^2 + Y^2$ when we change over to Cartesian coördinates—is unity). The equations of the other two circles are, similarly,

$$S + 2a_2x_2(\cot A_2 - \cot \theta) = 0, \qquad S + 2a_3x_3(\cot A_3 - \cot \theta) = 0;$$

and hence the radical center is given by the equations

$$a_1x_1(\cot A_1 - \cot \theta) = a_2x_2(\cot A_2 - \cot \theta) = a_3x_3(\cot A_3 - \cot \theta).$$

Or, since $\cot A_1 - \cot \theta = \sin (\theta - A_1)/\sin A_1 \sin \theta$, and $a_1/\sin A_1 =$ radius of circumcircle,

$$x_1 \sin (A_1 - \theta) = x_2 \sin (A_2 - \theta) = x_3 \sin (A_3 - \theta),$$

whence

$$x_1 : x_2 : x_3 = \frac{1}{\sin (A_1 - \theta)} : \frac{1}{\sin (A_2 - \theta)} : \frac{1}{\sin (A_3 - \theta)}.$$

If we wish to find the locus of the radical center we have

$$k/x_1 = \sin (A_1 - \theta) = \sin A_1 \cos \theta - \sin \theta \cos A_1,$$

and two similar equations where k is a factor of proportionality. These being linear and homogeneous in k, $\cos \theta$, $\sin \theta$ we have

$$\begin{vmatrix} \dfrac{1}{x_1} & \sin A_1 & \cos A_1 \\ \dfrac{1}{x_2} & \sin A_2 & \cos A_2 \\ \dfrac{1}{x_3} & \sin A_3 & \cos A_3 \end{vmatrix} = 0,$$

or expanded,

$$\frac{1}{x_1}\sin (A_2 - A_3) + \frac{1}{x_2}\sin (A_3 - A_1) + \frac{1}{x_3}\sin (A_1 - A_2) = 0;$$

and finally,

$$x_2x_3 \sin (A_2 - A_3) + x_3x_1 \sin (A_3 - A_1) + x_1x_2 \sin (A_1 - A_2) = 0.$$

It is immediately verifiable that this circumconic goes through the orthocenter (sec A_1, sec A_2, sec A_3) and the centroid (cosec A_1, cosec A_2, cosec A_3), but it is better to go back to the parametric equations of the curve. $\theta = 0°$ gives the centroid; $\theta = 90°$ gives the orthocenter. In addition $\theta = 60°$, $\theta = 120°$ give the two *Fermat Points, i.e.*, the sum of the distances of the point $\theta = 120°$ to the three vertices $A_1\ A_2\ A_3$ is a minimum.[1] It is left as an interesting exercise to prove that if on each segment A_2A_3, A_3A_1, A_1A_2 *isosceles* triangles with base angles θ are described such that when $0 < \theta < 90°$, each of the triangles has its third vertex on the *same* side of the segment as the remaining vertex of the original triangle, and when $90° < \theta < 180°$ on the *opposite* side, the joins of each vertex A to the vertex of the corresponding isosceles triangle are concurrent at the radical center discussed[2] above. In particular when the triangles are equilateral, *i.e.*, when $\theta = 60°$ or $120°$ we have the usual construction for the Fermat points.

It is now easy to follow the extension to the tetrahedron. Here we have four vertices A_1, A_2, A_3, A_4; four faces with areas a_1, a_2, a_3, a_4, circumcenters O_1, O_2, O_3, O_4 and circumradii R_1, R_2, R_3, R_4. What is to take the place of the circles of angle θ? A natural analogue of the circle 1 is a sphere through A_2, A_3, A_4 with the center at C_1 such that the angles $A_2C_1O_1 = A_3C_1O_1 = A_4C_1O_1 = \theta$. To find the equation of this sphere introduce, for the moment, rectangular axes (X, Y, Z) with origin at O_1 and Z axis along O_1C_1. The coördinates of the center C_1 are $(0, 0, R_1 \cot\theta)$, and the radius is $R_1 \operatorname{cosec}\theta$, so that the equation is $X^2 + Y^2 + Z^2 - 2R_1Z\cot\theta - R_1^2 = 0$. Now let A_1 denote the angle θ for the circumsphere so that the equation of the latter is $S \equiv X^2 + Y^2 + Z^2 - 2R_1Z\cot A_1 - R_1^2 = 0$ and we have as before, on introducing for the sake of symmetry tetrahedral coördinates (x_1, x_2, x_3, x_4), that the equations of the four spheres are $S + 2x_1R_1(\cot A_1 - \cot\theta) = 0$, etc., S being the equation of the circumsphere so written that its norm is unity. The radical center is accordingly given by

$$x_1 : x_2 : x_3 : x_4 = \frac{1}{\sin(A_1 - \theta)} : \frac{1}{\sin(A_2 - \theta)} : \frac{1}{\sin(A_3 - \theta)} : \frac{1}{\sin(A_4 - \theta)},$$

where we have availed ourselves of the fact that $R_1 = R\sin A_1$, R being the radius of the circumsphere. This is a curve through the vertices of the tetrahedron of reference; for when $\theta = A_1$ we get the vertex $x_1 : x_2 : x_3 : x_4 = 1 : 0 : 0 : 0$. Just as Kiepert's hyperbola degenerates into a line when the triangle is isosceles so this space curve will degenerate into a plane curve if any of the angles $A_1\ A_2\ A_3\ A_4$ are equal. It is at once apparent that an easy way to study the curve is to make use of the *focal transformation*[3]

$$y_1 = \frac{1}{x_1}; \quad y_2 = \frac{1}{x_2}; \quad y_3 = \frac{1}{x_3}; \quad y_4 = \frac{1}{x_4}.$$

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[1] Compare this MONTHLY, *1920*, 38–41.

[2] Kiepert, *l.c.*—EDITOR.

[3] This Cremona Transformation is of frequent occurrence and is called "inversion" by the French writers. However most mathematicians will agree that this word is rather overworked and the name used in the text is suggested. The points (x), (y) are the two foci of a quadric of revolution inscribed in the tetrahedron of reference.

Then the point y traces out a *line* which is the intersection of the two planes,

$$y_1 \sin (A_2 - A_3) + y_2 \sin (A_3 - A_1) + y_3 \sin (A_1 - A_2) = 0,$$

$$y_2 \sin (A_3 - A_4) + y_3 \sin (A_4 - A_2) + y_4 \sin (A_2 - A_3) = 0.$$

Now the transformation $y = 1/x$ sends an arbitrary plane into a cubic surface which meets a line in three points, and accordingly an arbitrary plane meets our curve in three points so that it is a *cubic* space curve. The point $\theta = 0$, which in the plane case gave the centroid, here gives the point $x_1 : x_2 : x_3 : x_4 = \operatorname{cosec} A_1 : \operatorname{cosec} A_2 : \operatorname{cosec} A_3 : \operatorname{cosec} A_4 = (1/R_1) : (1/R_2) : (1/R_3) : (1/R_4)$. The point $\theta = 90°$, which previously gave the orthocenter, is here $x_1 : x_2 : x_3 : x_4 = \sec A_1 : \sec A_2 : \sec A_3 : \sec A_4$. This point I propose to call the *Roberts's point,* since S. Roberts, in the *Educational Times,* 1887, problem 9093,[1] stated that the radical center of the four spheres, described on the faces as diametral planes, is the focal transform of the circumcenter of the tetrahedron. The straight line traced by the point y is accordingly described by saying that it passes through the circumcenter of the tetrahedron and the point whose coördinates are proportional to R_1, R_2, R_3, R_4.

That the curve whose parametric equations are

$$x_i = \operatorname{cosec} (A_i - \theta), \qquad i = 1, 2, 3, 4,$$

is a cubic can be seen easily in a more direct manner as follows: Eliminating θ from the three equations obtained by writing $i = 2, 3, 4$ in turn, we find

$$x_3x_4 \sin (A_3 - A_4) + x_4x_2 \sin (A_4 - A_2) + x_2x_3 \sin (A_2 - A_3) = 0.$$

This is a quadric cone with its vertex at the point (1, 0, 0, 0) and passing through the other vertices. Similarly the curve lies on the quadric cone

$$x_3x_4 \sin (A_3 - A_4) + x_4x_1 \sin (A_4 - A_1) + x_1x_3 \sin (A_1 - A_3) = 0.$$

These cones have a common generator—the join of the vertices A_1A_2 of the tetrahedron, and the remaining intersection is a cubic curve.

In conclusion one remark further may be made. It is a classical theorem on cubic space curves that, given four points on one, the cross ratio of the four planes, one through each of the points and a variable secant, is constant.[2] Taking the four vertices of the tetrahedron as the four fixed points, and the secant as passing through the *Roberts's* point and the point $(1/R_1, 1/R_2, 1/R_3, 1/R_4)$, it is not difficult to see that this cross ratio is

$$\sin (A_1 - A_3) \sin (A_2 - A_4)/[\sin (A_1 - A_4) \sin (A_2 - A_3)].$$

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[1] *Mathematical Questions and Solutions from the "Educational Times,"* vol. 48, 1883, pp. 37, 179.—EDITOR.

[2] Pascal, *Repertorium der höheren Mathematik,* vol. 2, 1902, p. 250.